New recipes for estimating default intensities

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- Default intensity model
- Pricing equation for CDS contracts
- Default intensity as solution of a Volterra equation of 2nd kind
- Comparison to common practise: The bootstrapping approach revisited
- Deriving default intensity from CDS spread observations. Special case of Nelson Siegel: Exact analytical solution and numerical recipe
- Simple approximation formula for default probabilities
- Fitting of stochastic models of default intensity: CIR Model

Credit Default Swap (CDS)

- *Reference firm*
  - Exposed to credit risk (which may include bankruptcy, failure to make principal or interest payments, restructering, …)
- *Protection buyer*
  - Periodically pays a premium (CDS-spread) until maturity or a credit event
- *Protection seller*
  - Provides protection from credit losses by paying a compensation in case of a credit event before maturity
- Contract details*:* Notional, Maturity, Payment frequency (e.g. quarterly), Credit event type, Settlement (physical / cash)

Premium contains information on market's estimate of credit risk.
Our goal is to extract this information from CDS spread observations.

Default Intensity Model

- Default probability (distribution of default time τ)

$$P(\tau \leq t) = 1 - \exp\left(-\int_0^t \lambda(u)du \right)$$

- pdf of τ

$$p(t) = \frac{dP(\tau \leq t)}{dt}$$

- *default intensity* (*default hazard rate)* λ is the conditional default arrival rate given no default up to time t:

$$\frac{1}{\Delta t} P(\tau \leq t + \Delta t \mid \tau > t) = \frac{1}{\Delta t} \frac{P(t < \tau \leq t + \Delta t)}{P(\tau > t)} \underset{\Delta t \to 0}{\longrightarrow} \lambda(t) \triangleq \frac{p(t)}{P(\tau > t)}$$

Valuation of risky cash flows

- Let the risk free rate (short rate) *r(t)* be given. Today's price of a riskless zero coupon bond with maturity *t* is then (under the assumption of an arbitrage free world)

$$D(t) = \exp\left(-\int_0^t r(u)du \right)$$

- More generally, suppose a riskless investment provides cash flows $C_{t_1},\ldots,C_{t_n}$ at times $t_1,\ldots,t_n$. Then the present value of this investment is

$$PV_{riskless}(C_{t_1},\ldots,C_{t_n}) = \sum_{i=1}^{n} C_{t_i} D(t_i)$$

- Now suppose that cash flows are contingent on the survival of an obligor with default time τ. (Their timing could also depend on τ.) Then the present value is

$$PV_{risky}(C_{t_1},\ldots,C_{t_n}) = \sum_{i=1}^{n} C_{t_i} D(t_i) I_{\tau > t_i}$$

where $I_{\tau > t}$ denotes the indicator function of the set (event) $\{\tau > t\}$.

Valuation of the Premium leg of a CDS

- The present value of premium payments

$$PV_{Premium} = \sum_{i=1}^{n} s \cdot \Delta t_i \cdot D(t_i) \cdot I_{\tau > t_i}$$

where s denotes the CDS spread and $\Delta t_i = t_i - t_{i-1}$ the time interval between premium payments and, as above, $I_{\tau > t_i}$ the survival indicator.

- By definition of the discount factors $D(t_i)$ and since $E(I_{\tau > t_i}) = P(\tau > t_i) = \exp\left(-\int_0^{t_i} \lambda(u) du \right)$

the expectation of the present value of the premium payments is

$$E(PV_{Premium}) = \sum_{i=1}^{n} s \cdot \Delta t_i \cdot \exp\left(-\int_0^{t_i} r(u) + \lambda(u) du \right)$$

Continuous premium payments

- With the notation $f(t) = \exp\left(-\int_0^t r(u) + \lambda(u) du \right)$ we get

$$E(PV_{\mathrm{Premium}}) = s \cdot \sum_{i=1}^{n} \Delta t_i \cdot f(t_i)$$

- For $\Delta t_i \to 0$ this may be approximated by

$$E(PV_{\mathrm{Premium}}) \approx s \cdot \int_0^T f(t) dt$$

- Using this approximation we implicitly assume continuous, accrued premium payments

Valuation of the default leg of a CDS

- The present value of the default payment is given by

$$PV_{Default} = \delta \cdot D(\tau) \cdot I_{\tau \le T}$$

where δ denotes the loss given default (fixed), T the maturity of the contract and $I_{\tau \le T}$ the default indicator.

- The expectation of the present value of default payments is given by

$$E(PV_{Default}) = \delta \cdot E(D(\tau) I_{\tau \le T}) = \delta \cdot \int_0^T D(t)\lambda(t)\exp\left(-\int_0^t \lambda(u)du\right)dt$$

With f defined as above (previous slide) this may be rewritten as

$$E(PV_{Default}) = \delta \cdot \int_0^T \lambda(t) f(t) dt$$

Pricing equation

- Under the above considerations the pricing equation

$$E(PV_{Premium}) = E(PV_{Default})$$

for the CDS contract becomes

(IEQ)

$$s(T) \cdot \int_0^T f(t)dt = \delta \cdot \int_0^T \lambda(t) f(t)dt$$

which is an integral equation for the function *f* . With the notation *s = s(T)* we emphazize the dependency of the CDS spread on the maturity *T*.

Piecewise constant default intensity

- In case of a constant deafult intensity the integral equation (IEQ) simplifies to

$$\lambda\big|_{[0,T]} = \frac{s(T)}{\delta}$$

- Now suppose the CDS spread observations $s(T_j)$ for different maturities $T_1 < T_2 < \ldots < T_m$ are given (e.g. $m = 5$, $T_1 = 1$, $T_2 = 3$, $T_3 = 5$, $T_4 = 7$ and $T_5 = 10$ years) and

$$\lambda(t) = \sum_{k=1}^{m} \lambda_k I_{(T_{k-1},T_k]}(t)$$

  where $T_0 = 0$. Then from (IEQ) applied to each spread observation we obtain a set of equations

$$s(T_k)\cdot\left[\sum_{j=1}^{k} J_j(\lambda_j)\cdot\exp\left(\sum_{i=1}^{j-1}(\lambda_{i+1}-\lambda_i)T_i\right)\right] = \delta\cdot\left[\sum_{j=1}^{k}\lambda_j J_j(\lambda_j)\cdot\exp\left(\sum_{i=1}^{j-1}(\lambda_{i+1}-\lambda_i)T_i\right)\right]$$

where $J_j(\lambda_j) = \int_{T_{j-1}}^{T_j} D(t)\cdot\exp(-\lambda_j\cdot t)dt$ (k =1,2,…,m) which may be solved iteratively for $\lambda_1,\ldots,\lambda_m$.

Piecewise constant default intensity (cont.)

- k = 1: derive λ_1 from equation

$$s(T_1) = \delta \cdot \lambda_1$$

- k = 2: derive λ_2 from λ_1 and equation

$$s(T_2) \cdot \left[\sum_{j=1}^{2} J_j(\lambda_j) \cdot \exp\left(\sum_{i=1}^{j-1} (\lambda_{i+1} - \lambda_i) T_i \right) \right] = \delta \cdot \left[\sum_{j=1}^{2} \lambda_j J_j(\lambda_j) \cdot \exp\left(\sum_{i=1}^{j-1} (\lambda_{i+1} - \lambda_i) T_i \right) \right]$$

- proceed like this until k = m.
- Disadvantage of this procedure (which is often referred to as „bootstrap“):
  - Default intensity discontinuous (as a step function)
  - No smoothing of noisy data

Solving the integral equation …

- Our approach relies on first smoothing CDS spread observations (e.g. by fitting a Nelson-Siegel type curve to the observations).

- Then we solve the integral equation (IEQ) for $f(t)$

- Once we have done that we may use the relation

$$(\log f(t))' = -(\lambda(t) + r(t)) \quad \text{to obtain} \quad \lambda(t) = -f'(t) / f(t) - r(t)$$

- Advantage of this approach:
  - Default intensity is a continuous function of time
  - Smoothing of noisy observations is done in a preceding step

Volterra equation

- Using the relation $\lambda(t) = -f'(t) / f(t) - r(t)$ equation (IEQ) may by put into the form

$$f(t) = 1 - \int_0^t \left(\frac{s(t)}{\delta} + r(x) \right) f(x) dx$$

  which is a Volterra equation of the 2nd kind.

- Instead of solving the integral equation directly we suggest to transform the problem to the problem of solving an ordinary linear differetial equation of 2nd order which is numerically easier to handle …

Differential equation

- Tranformation into an ordinary linear differential equation of 2nd order:

(ODE) $$f''+f'\cdot g+f\cdot h=0$$

with time dependend coefficients

$$g(t)=r(t)+\frac{s(t)}{\delta}+\frac{s''(t)}{s'(t)}$$ and $$h(t)=r'(t)+\frac{2s'(t)}{\delta}-\frac{s''(t)}{s'(t)}\left(r(t)+\frac{s(t)}{\delta}\right)$$

where

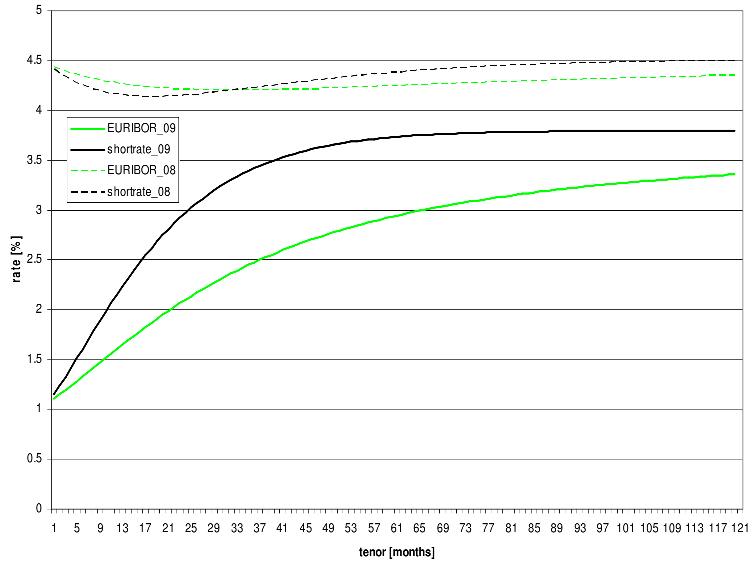
- *s(t)* : CDS spread curve
- *r(t)* : reference curve (*risk free rate*)

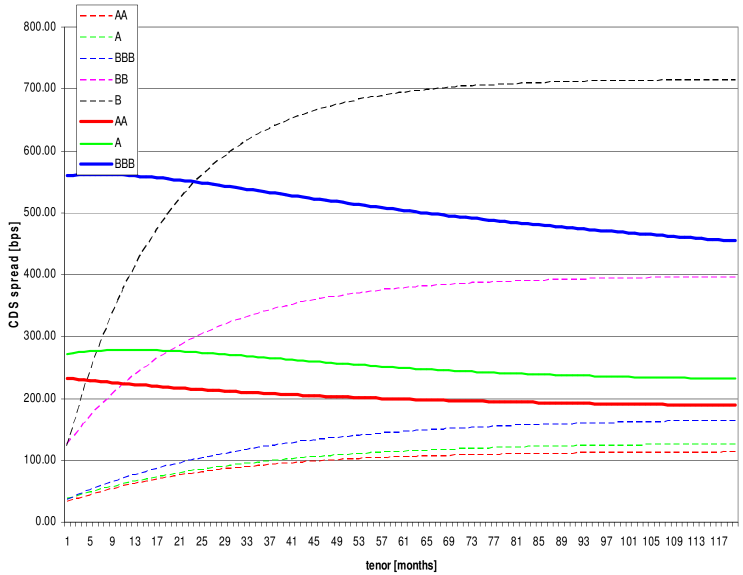
with initial conditions: $f(0)=1$ and $f'(0)=-\lambda(0)-r(0)$

where $$\lambda(0)=\frac{s(0)}{\delta}$$

Special case of Nelson Siegel: Exact analytical solution and numerical recipe

- NS curves are fitted to the CDS quotes as well as the short rates





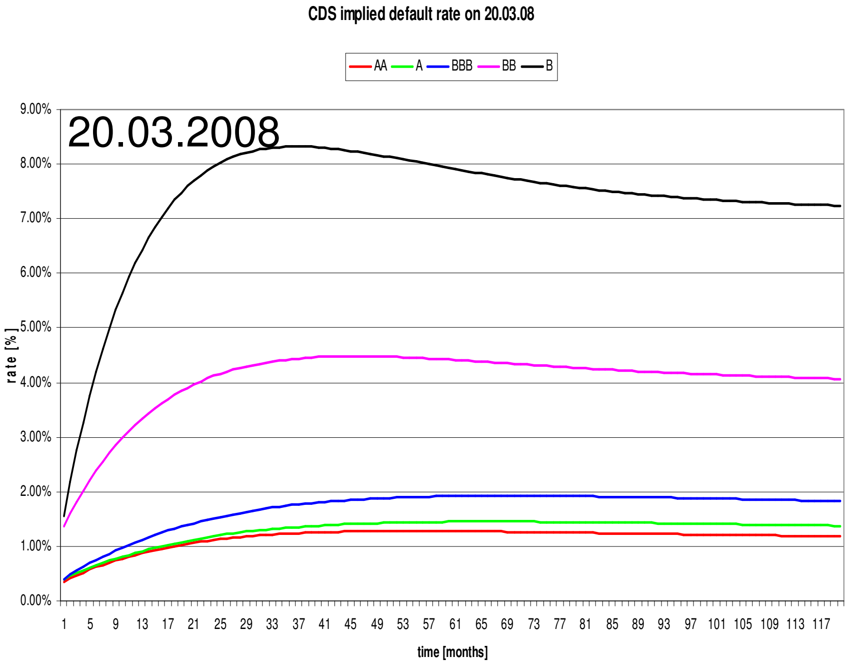
- the fitted curves in a form
  - short rate on 20.03.08: $r(t) = 0.045 - (0.000613 + 0.0059\ t)\ \exp(-0.617\ t)$
  - short rate on 20.03.09: $r(t) = 0.037 - (0.026 + 0.017\ t)\ \exp(-1.032\ t)$
  - CDS spread $s(t) = a - (b + c\ t)\ \exp(-d\ t)$

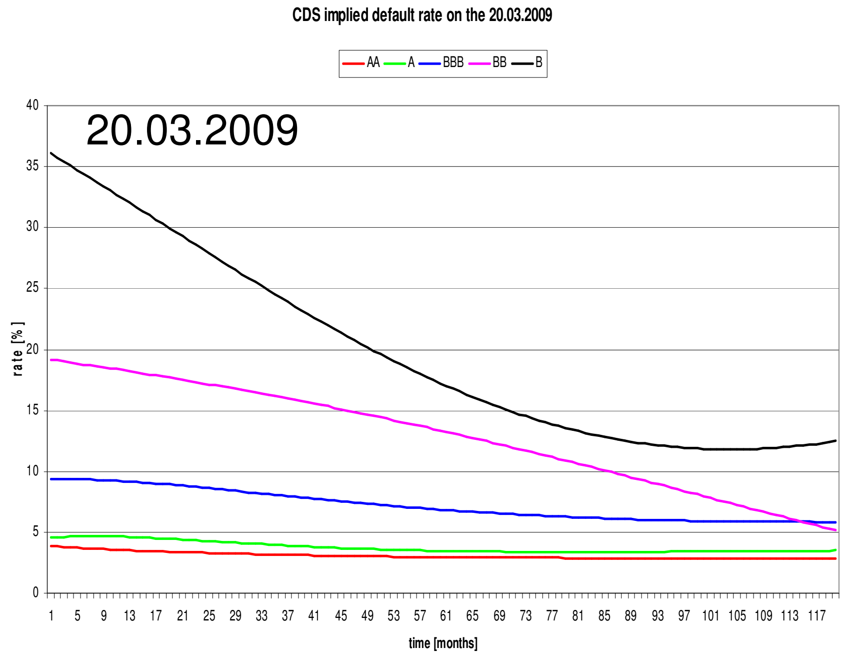
Implied default intensity term structure

- the fitted curves s(t) and r(t) as well as LGD δ =0.6 as an input of the ODE

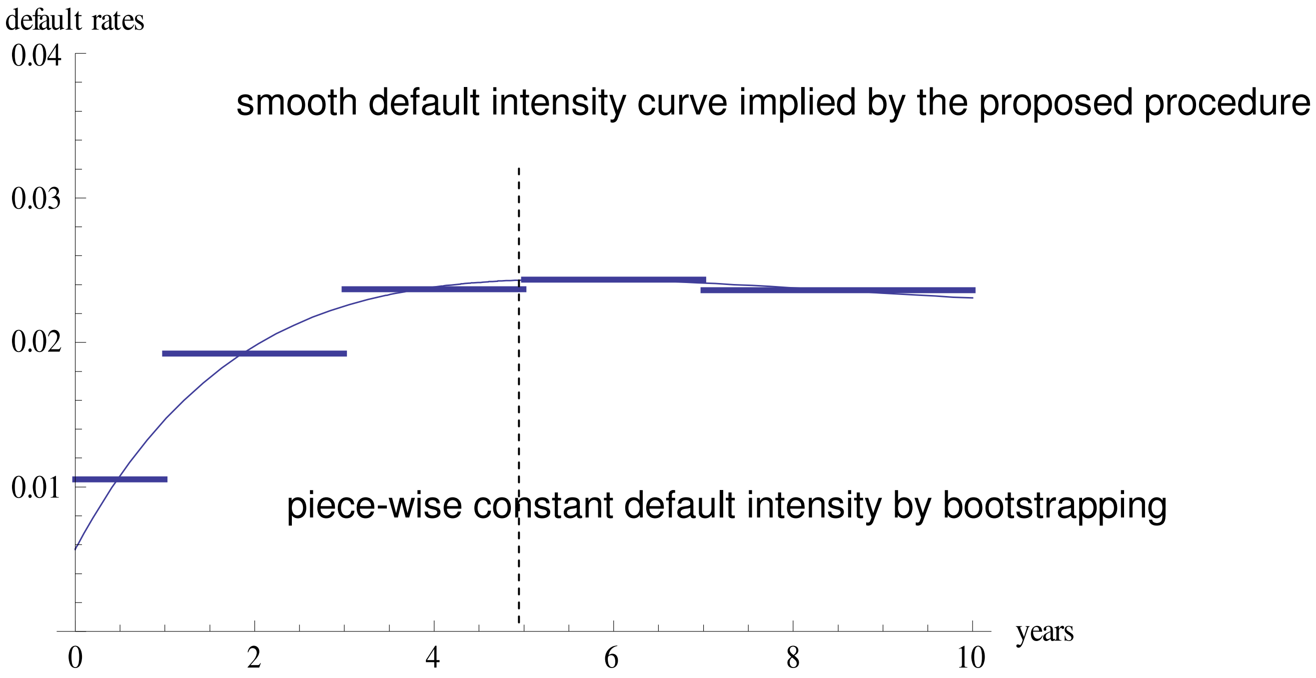
$$f''+f'\cdot g+f\cdot h=0$$

- default intensity $\lambda(t)=-\frac{f'}{f}-r(t)$ is an output of the ODE





Comparison to bootstrapping approach

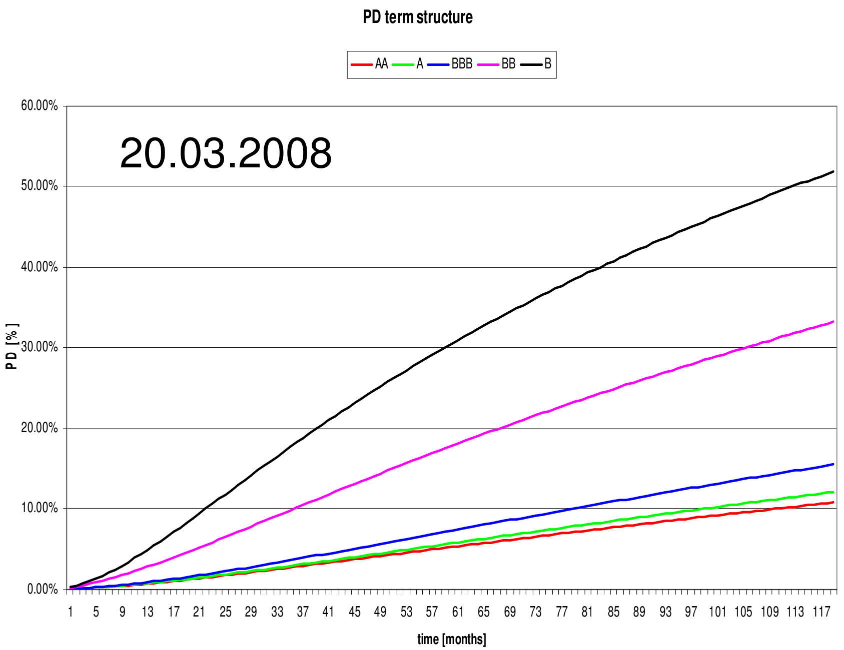


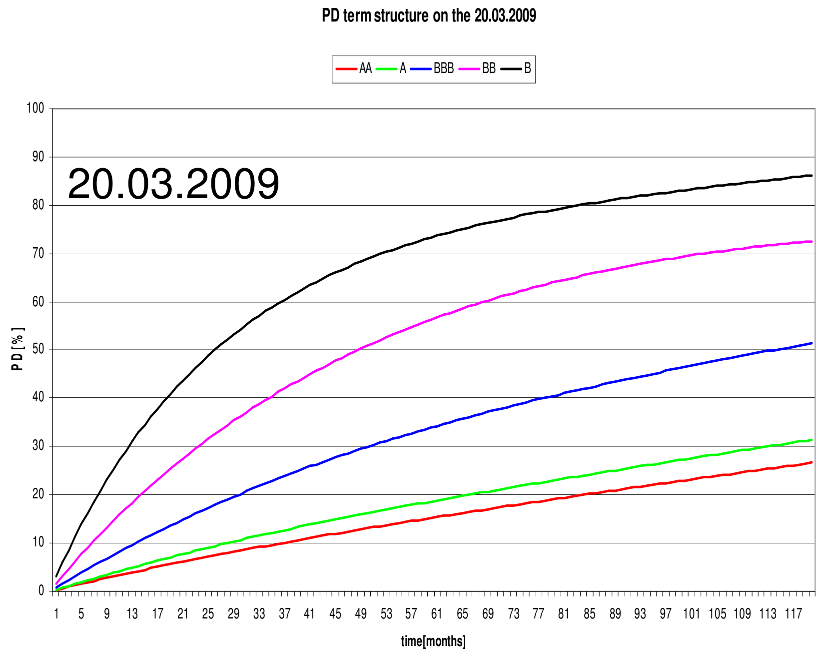
Default intensity for European A-rated corporates as of as of 20.03.2008

for maturities below 5 years:
piece-wise constant default intensity displays an economically unintuitive behaviour (large jump sizes) and is potentially vulnerable to anomalies in the data (zig-zag behaviour).
These disadvantages are avoided in the Nelson Siegel fitting approach.

Implied PD term structure

- *PD* $$PD(t) = P(\tau \le t) = 1 - \exp\left(-\int_0^t \lambda(u)du\right)$$





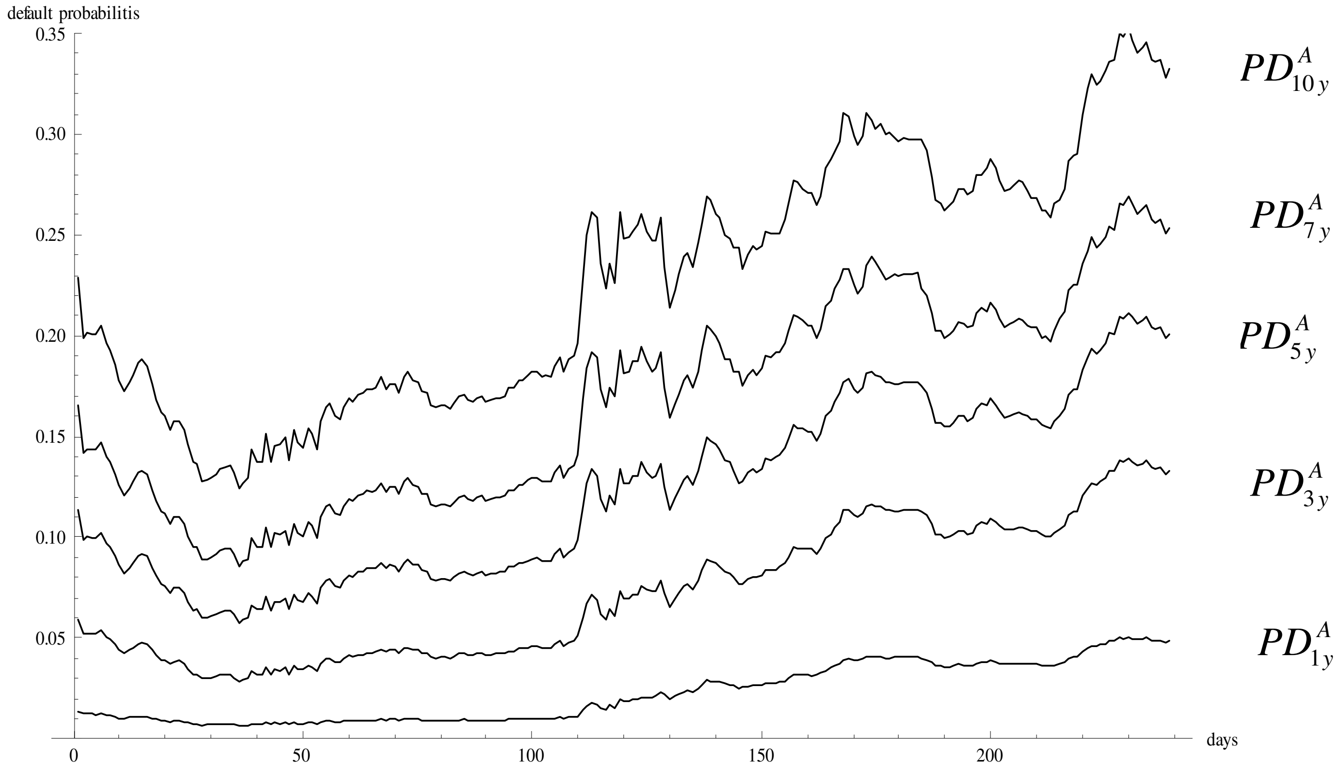
- default rate and PD can be done in a closed form

$$PD(t, 20.03.2008) = 1 - e^{(\bar{r}-\gamma)\cdot t - 0.025\cdot\nu(t)}\left(\nu(t)^{0.152}\left[-0.166\cdot + \frac{0.995}{\nu(t)}\right] + 0.197\cdot e^{0.025\cdot\nu(t)}\,{}_1F_1(2, 1.85, -0.025\nu(t))\right)$$

$\gamma = 0.369;\ \nu(t) = \exp(-\gamma\cdot t),\ \bar{r} = 4.34\%;\quad {}_1F_1(a.b; z)$ is the Kummer confluent hypergeometric function

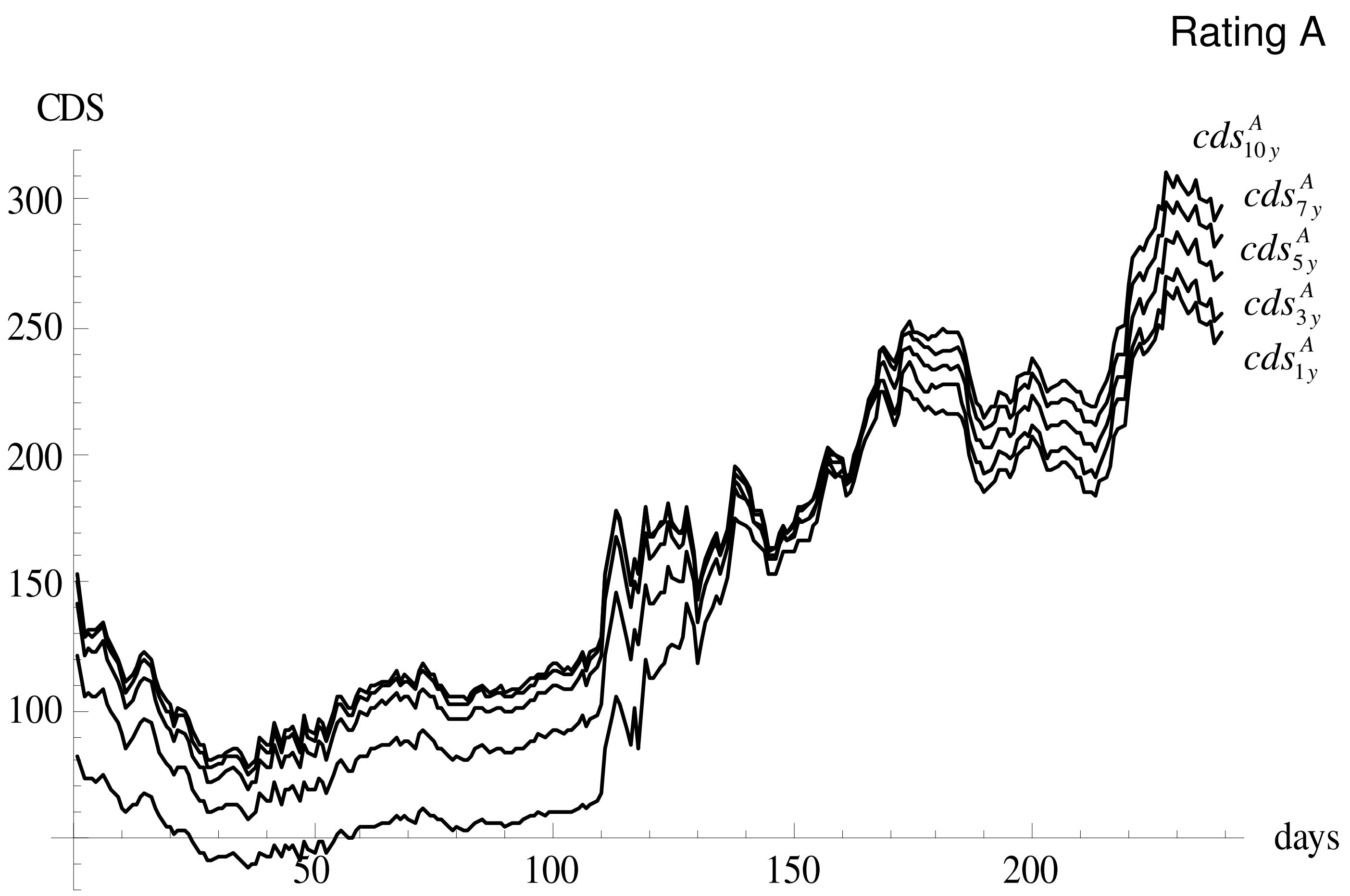
Implied term-structure of cumulative default probabilities for A-rated corporates

Rating A



20.03.08 < *τ (days)* < 20.03.09

A-rated CDS term-structure



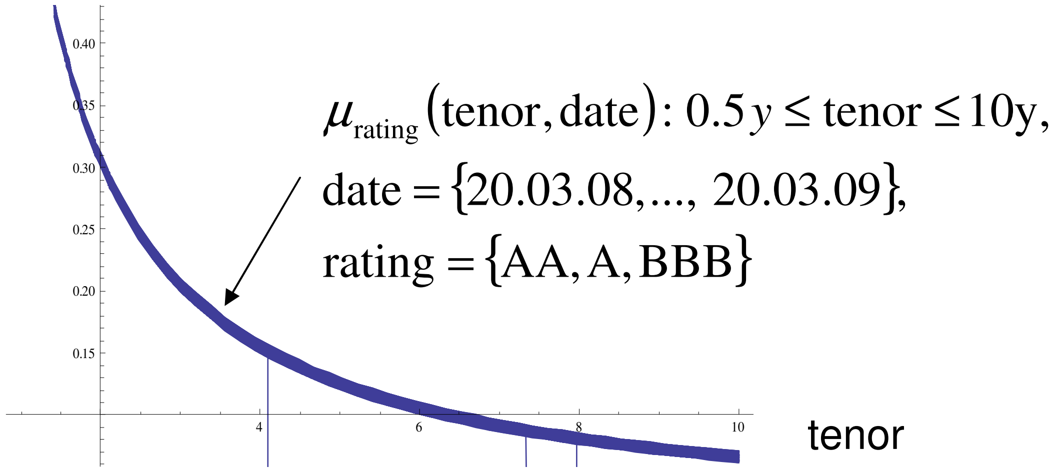
20.03.08 < *τ (days)* < 20.03.09

Simple approximation formula for default probabilities

We construct a ratio

$$\mu_{\text{rating}}(\text{tenor},\text{date}) = \frac{CDS_{rating}(\text{tenor},\text{date})}{PD_{rating}(\text{tenor},\text{date})}$$

stable over time for a large range of rating classes !!!



$$PD_R(t,x) = \left(0.065 + 0.468e^{-0.039 \cdot t} + 0.835e^{-0.222 \cdot t}\right)^{-1} CDS_R(t,x),$$
$$\forall x,\ R = AA \vee A \vee BBB,\ 0.5 \le t\,[years] \le 10$$

Fitting of stochastic models of default intensity: CIR Model-I

We consider a case of a stochastic default intensity

$$d\lambda = \alpha \cdot (\theta - \lambda) dt + \sigma \sqrt{\lambda} dW$$

before: $f(t) = \exp\left(-\int_0^t r(u) + \lambda(u) du \right)$

now: $f(t) = E\left(\exp\left(-\int_0^t r(u) + \lambda(u) du \right) \right)$

⇩

$$f_{CIR}(t) = D(t) \cdot E\left[\exp\left(-\int_0^t \lambda(u) du \right) \middle| \lambda(0) \right] \equiv D(t) \cdot \exp(-A(t) \cdot \lambda(0) + C(t))$$

Affine term structure

$$A(t) = \frac{2}{(\beta + \alpha) + 2\beta\left(e^{\beta \cdot t} - 1\right)^{-1}} \qquad C(t) = \frac{2\alpha\theta}{\sigma^2} \cdot \ln\left[\frac{2\beta \cdot e^{(\beta+\alpha)\cdot t/2}}{(\beta + \alpha) \cdot \left(e^{\beta \cdot t} - 1\right) + 2\beta} \right]$$

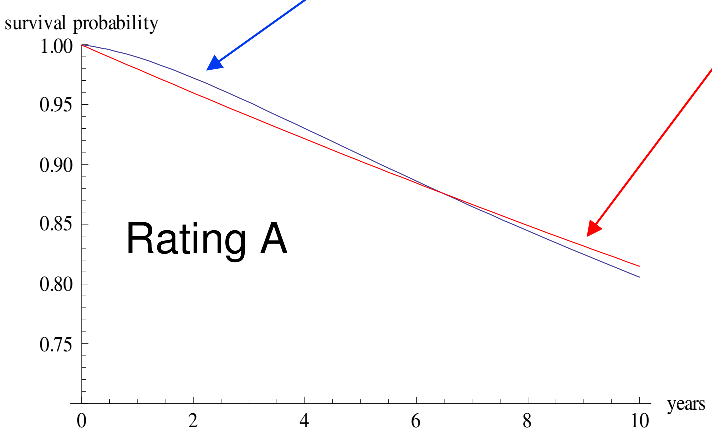
$$\beta = \sqrt{\alpha^2 + 2\sigma^2}$$

Fitting of stochastic models of default intensity: CIR Model-II

$$f_{CIR}(t) = D(t)\cdot\exp(-A(t)\cdot\lambda(0) + C(t))$$

solves $f'' + f' \cdot g + f \cdot h = 0$ with the input parameters s(t) (CDS), r(t) (short rate), δ (LGD)

$$\sum_i \left[\underbrace{f(t_i)\cdot D^{-1}(t_i)}_{1-PD(t_i)} - \underbrace{\exp\left(-A(t_i)\cdot s(0)\cdot\delta^{-1} + C(t_i)\right)} \right]^2 \to \min_{\{\alpha,\theta,\sigma\}}$$



$\alpha = 2.74$, $\sigma = 450.73$ and $\theta = -2.38$ on 20.03.2008

Conclusion

- New method for deriving default intensity from CDS spread observations

The method has two main advantages:

- The default intensity naturally becomes a continuous function of *t* and no economically unintuitive discontinuities arise.
- The procedure is stable w.r.t. outliers and noisy data (e.g. due to erroneous CDS-quotes) because is relies on a preceding smoothing procedure.

The new estimation procedure also serves as a stable basis for fitting stochastic default intensity models like CIR.