# State-dependent dependencies

## A continuous-time dynamics for correlations

Christoph Becker and Wolfgang M. Schmidt

Frankfurt School of Finance & Management
Centre of Practical Quantitative Finance

Eurobanking 2010

## Asset Dependencies in Finance are critical for

- *Asset allocation & portfolio optimisation*
  (diversification, Markowitz' theory)
- *Portfolio risk management*
  (portfolio risk measured by variance, value-at-risk, expected shortfall etc.)
- *Multi-asset options pricing & hedging*
  (basket options, index versus single asset options, **interest rate swaptions**→ cap & swaption puzzle . . . )

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## Dependencies in Financial Economics

- *Contagion of financial markets*
  (Propagation of shocks in a crisis, cross-country dependencies)
- *Impact of asset dependencies on real economy*
  (Integration of capital markets and economic growth, globalisation)

The strength of dependence is traditionally measured by correlation or some copula based measure.
Correlation $\rho$ between log-returns of assets $S^1, S^2$:

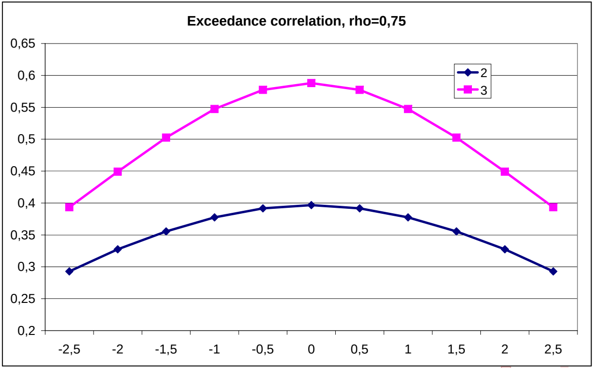
$$\rho = \text{Corr}\left(\log\left(\frac{S^1(t+\Delta)}{S^1(t)}\right), \log\left(\frac{S^2(t+\Delta)}{S^2(t)}\right)\right).$$

**But dependencies between assets are not constant but vary over time and are likely to depend on the state of the market:**
*"In bear markets or a crisis correlation tends to be higher than normal."*

To detect whether correlation between to random variables $X, Y$ is state dependent or not, the naive approach is to look at *exceedance correlation*

$$\rho(a,b) = \text{Corr}\left(X, Y \middle| X \in [a,b], Y \in [a,b]\right).$$

But even for normals $X, Y$ with fixed correlation $\rho$, the exceedance correlation is not constant: [Campell et al., 2008]



Approaches to non-constant correlation in the literature are, for example, based on

- instantaneous correlation follows own stochastic dynamics (like Heston),
- regime shift approaches,
- explicit correlation behaviour in different market environments (normal, bear etc) $\rightarrow$ requires subjective definition of appropriate environments.

**We propose a continuous-time dynamics for correlation driven intrinsically and continuously by the state of the market.**

We describe the market state by a function $F(t,S)$ that depends on time $t$ and the vector $S = (S^1, \ldots, S^N)$ of asset paths. An example for the market state at time $t$ is the realised market trend over a recent time window $[t-r,t)$.

Correlation is assumed to be a function of the market state

$$\boxed{\rho = \rho(..., F(t,S)).}$$

Our model setup is a system of stochastic differential equations with delay:

$$\begin{aligned} \mathrm{d}\, S^i(t) &= \mu_i(\theta) S^i(t)\,\mathrm{d}t + \sigma_i\Big(\theta, F(t,S)\Big) S^i(t)\,\mathrm{d}W^i(t) \\ \mathrm{d}[W^i, W^j]_t &= \rho_{ij}\Big(\theta, F(t,S)\Big)\,\mathrm{d}t \end{aligned}$$

Volatilities $\sigma_i(\theta,\cdot)$ and correlation $\rho_{ij}(\theta,\dot{)}$ are parameterised functions of the market state.

## Proposition (Existence of Model)

*Under standard smoothness conditions there exists a unique solution $S$ with strictly positive paths for every $\theta \in \Theta$.*

Interpretation of the correlation $\rho$

$$\lim_{\Delta\to 0} \mathrm{Corr}\left(\log\left(\frac{S^1(t+\Delta)}{S^1(t)}\right), \log\left(\frac{S^2(t+\Delta)}{S^2(t)}\right)\middle|\mathscr{F}_t\right) = \rho\Big(\theta, F(t,S)\Big).$$

The functional form for the market state

$$F : [0,T] \times \mathbb{C} \to \mathbb{R}$$

and the volatilities and instantaneous correlations

$$\begin{aligned} \sigma_i(\theta,\cdot) &: \quad \mathbb{R} \to \mathbb{R} \\ \rho(\theta,\cdot) &: \quad \mathbb{R} \to [-1,1] \end{aligned}$$

need to be specified. Our proposal has been motivated by discrete time analyses.

## Definition (Market state function $F$)

*$F$ is the realised drift over $n_F$ past days averaged over all assets. Set $\Delta = 1$ day.*

$$F(t,S) = \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\nu}(t,S^i) + \frac{1}{2}\widehat{\sigma^2}(t,S^i)\right),$$

*where*

$$\widehat{\nu}(t,S^i) = \frac{1}{\Delta n_F}\sum_{k=1}^{n_F}\log\frac{S^i(t-(k-1)\Delta t)}{S^i(t-k\Delta)} \quad \text{“which is like } \mu - \frac{1}{2}\sigma^2\text{”}$$

$$\widehat{\sigma^2}(t,S^i) = \frac{1}{\Delta(n_F-1)}\sum_{k=1}^{n_F}\left(\log\frac{S^i(t-(k-1)\Delta t)}{S^i(t-k\Delta)} - \widehat{\nu}(t,S^1)\right)^2.$$

Motivated by discrete time analysis we propose the parameterisations

- $$\rho(\theta, F) := \frac{2}{\pi} \arctan\left(h_{(\xi,\eta)}(F)\right),$$

  with $h_{(\xi,\eta)}$ cubic spline through $\theta = (\xi_i, \eta_i)_{i=1,\dots,n_\rho}$ and $(\xi_i)_i$ equidistant points on empirically observed $[\min_t F(t,S), \max_t F(t,S)]$.
- analogously for $\sigma$,

  $$\sigma(\theta, F) = |g_{(\xi,\eta)}(F)|.$$

How to estimate the model parameters $\theta$ from (daily) observations $(s(t_k))_{k=1,\dots,n}, t_k = k\Delta$?

Statistical methods for stochastic delay differential equations not yet well-developed.

Consider the likelihood function

$$f_{\left(\theta,\log S(t_{1+n_F}),\ldots,\log S(t_n)\right)}\Big(\log s(t_{1+n_F}),\ldots,\log s(t_n)\Big)$$
$$= \prod_{i=1+n_F}^{n} p_i\left(\log s(t_i)|\log s(t_1),\ldots,\log s(t_{i-1}),\theta\right),$$

with $p_i$ the (in our case unknown!) conditional transition densities of $\log S(t_i)$ given $\log S(t_1),\ldots,\log S(t_{i-1})$.

**Proposed Maximum-Likelihood Estimator:**

Approximate $p_i$ with density $\widehat{p_i}$ of a $N$-dimensional normal distribution with parameters motivated by the Euler scheme, see [Küchler et al., 2000], [Baker and Buckwar, 2000]. Our ML-estimator is

$$\widehat{\theta}_n = \operatorname{argmax}_{\theta\in\Theta} \sum_{i=1+n_F}^{n} \log\widehat{p_i}\left(\log s(t_i)|\log s(t_1),\ldots,\log s(t_{i-1}),\theta\right).$$

Typical shapes of correlation and volatility as function on market state ($n_F = 50$).

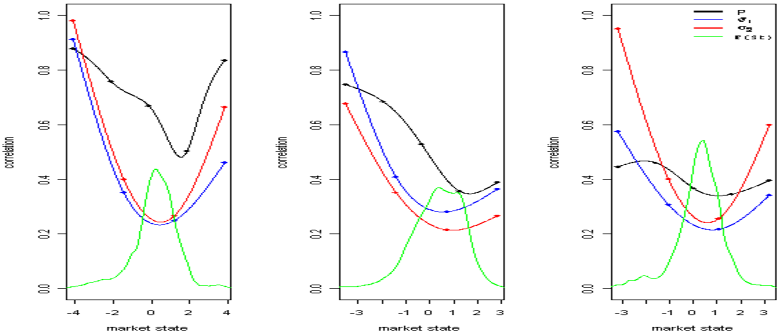



Figure: Estimations for Commerzbank - Deutsche Bank, BASF-VW, Commerzbank - EON, 1990 – 2008.

Determine optimal length $n_F$ of market memory:

$$\widehat{n_F} = \text{argmax}_{n_F} \max_{\theta \in \Theta} \log f_{\left(\theta, \log S(t_{1+n_F}), \ldots, \log S(t_n)\right)}\Big(\log s(t_{1+n_F}), \ldots, \log s(t_n)\Big)$$

Approach is motivated by model selection criteria like Akaike's Information Criterion.

# Estimate optimal market memory $n_F$

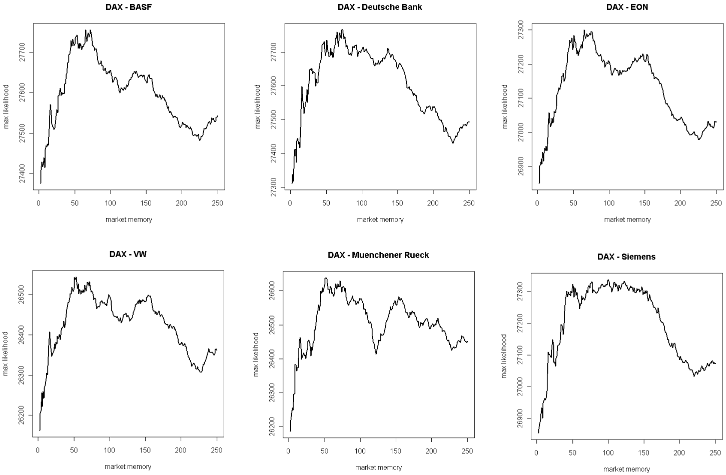



Figure: Estimations based on daily observations Jan. 1990 – May 2008.

US stock market: typical shapes of correlation and volatility (market state determined by the S&P 500, $n_F = 75$).

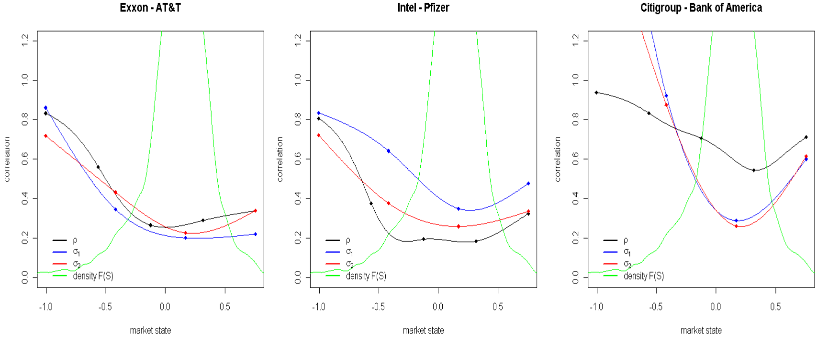



Figure: Estimations based on daily observations, Jan. 1990 – March 2010.

# Deviation of market state dependent correlation from their averages $\rightarrow$ contagion

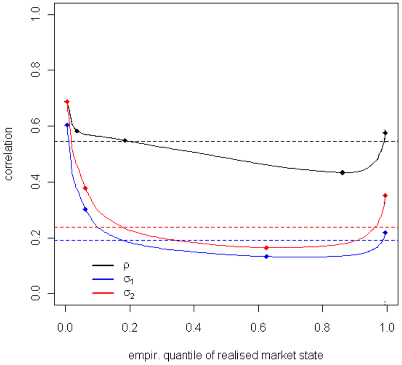
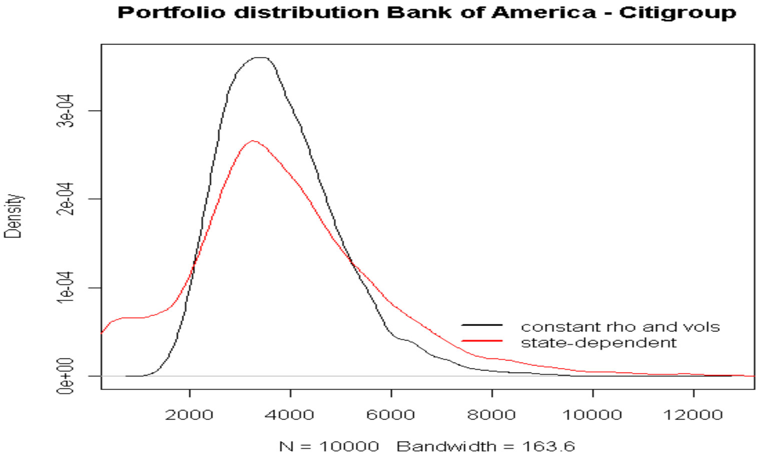



Figure: Correlation risk for DAX - S&P 500, 1990 – 2009, converted to EURO, $n_F = 92$ days.

**Portfolio Risk:** For models with constant and models with state dependent dependencies and volatilities the profit & loss distribution of a portfolio is different. This impacts portfolio risk measures.



**Option Pricing:** Price and hedge option with payoff $h(S^1(T), \ldots, S^n(T)) \geq 0$.

True asset dynamics for $S = (S^1, \ldots, S^n)$ under $\mathbf{Q}_{\text{true}}$:

$$\begin{aligned} \mathrm{d}S^i(t) &= r\,S^i(t)\,\mathrm{d}t + \sigma_{\text{true},i}\big(F(t,S)\big)\,S^i(t)\,\mathrm{d}W^{i,\mathbf{Q}_{\text{true}}} \\ \mathrm{d}\,[W^{i,\mathbf{Q}_{\text{true}}}, W^{j,\mathbf{Q}_{\text{true}}}]_t &= \rho_{\text{true},i,j}\big(F(t,S)\big)\,\mathrm{d}t \end{aligned}$$

Misspecified dynamics under measure $\mathbf{Q}_{\text{false}}$:

$$\begin{aligned} \mathrm{d}S^i(t) &= r\,S^i(t)\,\mathrm{d}t + \sigma_{\text{false},i}\big(t,S(t)\big)\,S^i(t)\,\mathrm{d}W^{i,\mathbf{Q}_{\text{false}}} \\ \mathrm{d}\,[W^{i,\mathbf{Q}_{\text{false}}}, W^{j,\mathbf{Q}_{\text{false}}}]_t &= \rho_{\text{false},i,j}\big(t,S(t)\big)\,\mathrm{d}t \end{aligned}$$

Fair price under model $\mathbf{Q}_{\text{false}}$ is

$$V(t, x_1, \ldots, x_n) = e^{-r(T-t)} \mathbb{E}_{\mathbf{Q}_{\text{false}}} \left( h(S^1(T), \ldots, S^n(T)) \middle| S^1(t) = x_1, \ldots, S^n(t) = x_n \right).$$

Hedging strategy $\Delta$ based on $\mathbf{Q}_{\text{false}}$ with value process

$$\Pi_\Delta(t) = \beta(t) e^{rt} + \sum_{i=1}^{n} \partial_{x_i} V\left(t, S^1(t), \ldots, S^n(t)\right) S^i(t).$$

Problem: How reliable are the price $V$ and hedge $\Pi_\Delta$ derived from a wrong model dynamics?

## Proposition (adapting ideas from [El Karoui et al., 1998])

*If for $\mathrm{d}t \otimes \mathbf{Q}_{true}$-almost all $(t,\omega)$ and for all $i,j = 1,\dots,n$*

$$\Big[\rho_{false,i,j}(t,S(t))\sigma_{false,i}(t,S(t))\sigma_{false,j}(t,S(t)) - \rho_{true,i,j}(F(t,S))\sigma_{true,i}(F(t,S))\sigma_{true,j}(F(t,S))\Big] \times \frac{\partial^2}{\partial x_i \partial x_j} V\left(t, S^1(t),\dots,S^n(t)\right) \geq 0,$$

$$\frac{\partial^2}{\partial {x_i}^2} V(t,S^1(t),\dots,S^n(t))\left(\sigma^2_{false,i}(t,S(t)) - \sigma^2_{true,i}(F(t,S))\right) \geq 0,$$

*then the price under the wrong model overestimates the true price,*

$$\boxed{\mathrm{e}^{-rT}\mathbb{E}_{\mathbf{Q}_{false}}\left(\frac{h(S^1(T),\dots,S^n(T))}{B(T)}\right) \geq \mathrm{e}^{-rT}\mathbb{E}_{\mathbf{Q}_{true}}\left(\frac{h(S^1(T),\dots,S^n(T))}{B(T)}\right),}$$

*and the hedge computed in the wrong model is a super-hedge under the true asset dynamics, i.e.,*

$$\boxed{\Pi_\Delta(T) \geq h(S^1(T),\dots,S^n(T)), \qquad \mathbf{Q}_{true} - a.s.}$$

## Risk management

Consider a portfolio with $n$ assets

$$\sum_{i=1}^{n} \alpha_i S^i(t), \qquad t \in [0, T]$$

and weights $\alpha_i \in \mathbb{R}^+$. Are risk measures like the portfolio variance robust with respect to misspecified volatilities and correlations?

### Proposition

*If*

$$\sigma_{false,i} \geq \sigma_{true,i}\Big(F(t,S)\Big), \qquad \rho_{false,i,j} \geq \rho_{true,i,j}\Big(F(t,S)\Big) \vee 0, \qquad \mu_{false,i} \geq \mu_{true,i}(t),$$

*then the false model overestimates the true portfolio risk, that is,*

$$\boxed{\mathrm{Var}_{\mathbf{P}_{false}}\left(\sum_{i=1}^{n} \alpha_i S^i(T)\right) \geq \mathrm{Var}_{\mathbf{P}_{true}}\left(\sum_{i=1}^{n} \alpha_i S^i(T)\right).}$$

Ang, A. and Chen, J. (2002).
Asymmetric correlations of equity portfolios
*Journal of Financial Economics*, 63(3), 443–494

Baker, C. and Buckwar, E. (2000).
Numerical analysis of explicit one-step methods for stochastic delay differential equations.
*LMS J. Comput. Math*, 3:315–335.

Campbell, R.A.J. and Forbes, C.S. and Koedijk, K.G. and Kofman, P. (2008).
Increasing correlations or just fat tails?
*Journal of Empirical Finance*, 15 (2), 287–309

El Karoui, N., Jeanblanc-Picque, M., and Shreve, S. (1998).
On the robustness of the black-scholes equation.
*Mathematical Finance*, 8:93–126.

Küchler, U., Platen, E., and Group, Q. F. R. (2000).
*Strong discrete time approximation of stochastic differential equations with time delay*.

School of Finance and Economics, University of Technology, Sydney.