EUROBANKING 2010

Derivation of Continuous, Real and Nominal Yield Curves for the Israeli Government Bond Market, Using The Nelson-Siegel-Svensson Type Curves

Alex Arensburg

Bank Hapoalim, Tel Aviv, ISRAEL

May 2010

Overview

1. The ISRAELI Government Bond Market (in ILS)
2. The Nelson-Siegel-Svensson (**NSS**) family of curves
3. Selecting “the best” NSS curve using bonds market data: a robust optimization procedure.
4. Nominal and real yield curves - some examples
5. Conclusions

The Market for Real and Nominal Government Bonds in Israel

As of January 2010:

Sector	Shekel Nominal	Shekel Linked
No. of Bonds	24	17
Longest Maturity	Oct-2026	May-2036
Total Face Value (M-ILS)	225,100	122,000
% of total	65	35
% of GDP	31.6	17.1

The Nelson-Siegel-Svensson Curves

The NSS curves are a six parameters family of continuous curves given by:

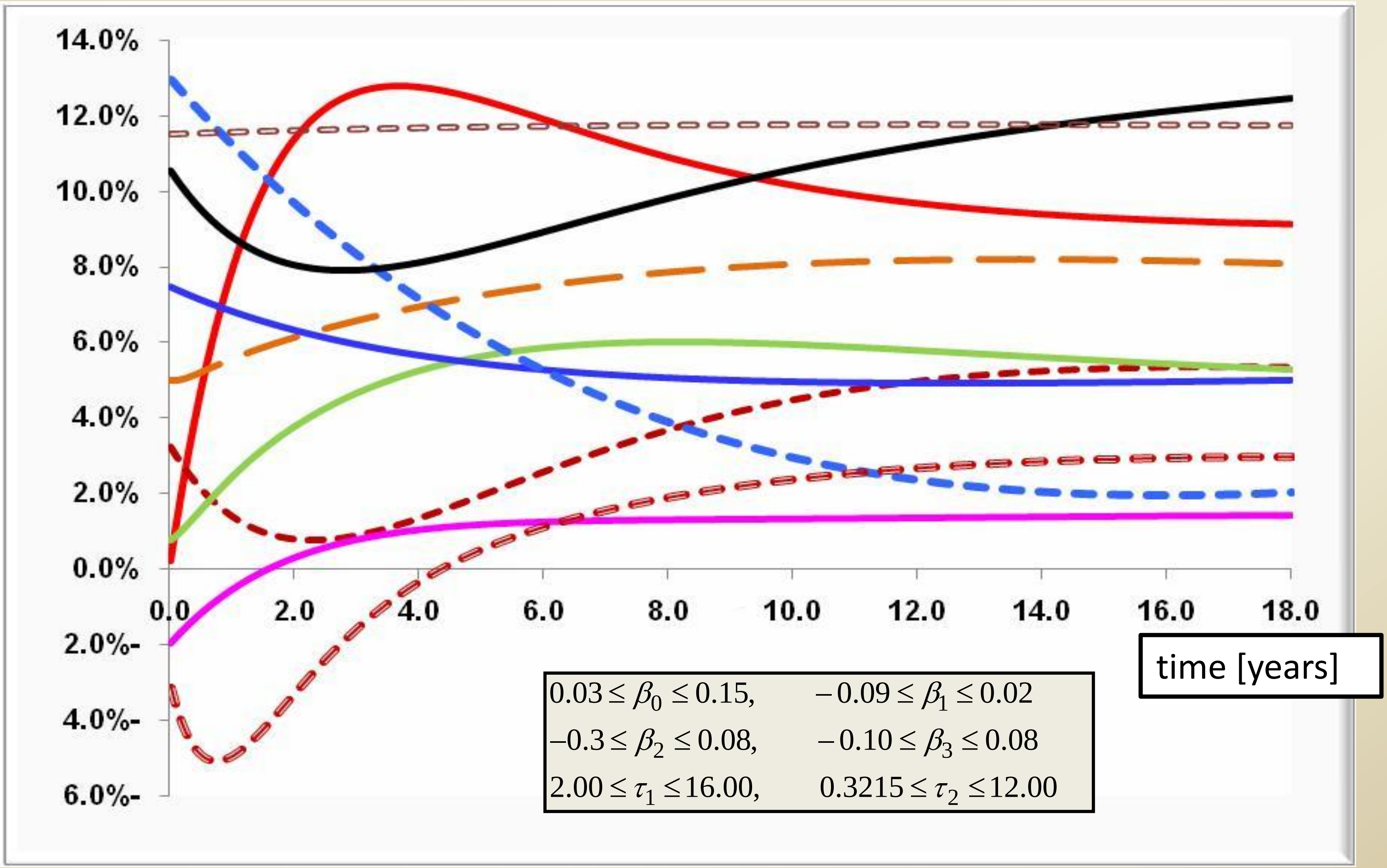
$$f(\beta_0,\beta_1,\beta_2,\beta_3,\tau_1,\tau_2,t) = \beta_0 + \left(\beta_1 + \beta_2 \frac{t}{\tau_1}\right) e^{-t/\tau_1} + \beta_3 \frac{t}{\tau_2} e^{-t/\tau_2}$$

The function $f(\beta,\tau,t)$ can be used to model the term structure of interest rates, $r(t)$ (or forward rates).

Notation:

$$r_{NSS}(t) \equiv f(\beta_0,\beta_1,\beta_2,\beta_3,\tau_1,\tau_2,t) \equiv f(\boldsymbol{\beta},\boldsymbol{\tau},t)$$

Some Shapes of NSS Curves



Fitting NSS Curves

Standard technique:

Minimize the mean square difference between market prices and prices calculated using the NSS curve with respect to $\boldsymbol{\beta}$ and $\boldsymbol{\tau}$

Denote by P_m^k the market price of Gov. bond k and by $P_{NSS}^k(\boldsymbol{\beta},\boldsymbol{\tau})$ the price that results from discounting the (bond) cash flow using an NSS curve.

$$P_{NSS}^k(\boldsymbol{\beta},\boldsymbol{\tau}) = \left(\frac{CPI_{known}}{CPI_{base}}\right)\sum_{i=1}^{N}\frac{CF_i^k}{\left(1+r_{NSS}(\boldsymbol{\beta},\boldsymbol{\tau},T_i)\right)^{T_i}}$$

$$r_{NSS}(\boldsymbol{\beta},\boldsymbol{\tau},T) = \beta_0 + \left(\beta_1 + \beta_2\frac{T}{\tau_1}\right)e^{-T/\tau_1} + \beta_3\frac{T}{\tau_2}e^{-T/\tau_2}$$

Fitting NSS Curves

Define the target function to minimize with respect to $\boldsymbol{\beta}$ and $\boldsymbol{\tau}$ as:

$$S_P(\boldsymbol{\beta},\boldsymbol{\tau}) = \sum_{k=1}^{K} W_k \left(P_m^k - P_{NSS}^k(\boldsymbol{\beta},\boldsymbol{\tau}) \right)^2$$

The weights W_k are sometimes taken as the $W_k=1/D_k$ where D_k is the duration of bond k. It may also be necessary to impose some constraints such as:

$$\tau_i \ge \tau_i^* \ (=0)$$
$$\beta_0 + \beta_1 = r_a \quad \& \quad \beta_0 > 0$$
$$\left| r_{NSS}(t_a) - r_{NSS}(t_b) \right| \le X\%$$

Fitting NSS Curves

Alternative approach

Representation of the bond market, definitions and notation:

- **The cash flow matrix** $CF_{i,k} \quad k=1,\ldots,K \qquad i=1,\ldots,N$
- **The timing of cash flows** $t_i \qquad i=1,\ldots,N$
- **The Discrete Yield Curve (DYC) vector** $R_i \qquad i=1,\ldots,N$
- **The base index for the ILBs** $CPI_0^k \qquad k=1,\ldots,K$
- **The market prices of bonds** $P_m^k \qquad k=1,\ldots,K$

Matrix representation (in Excel)

t_i {years} $\qquad k \rightarrow \qquad R_i$

$i \downarrow \qquad CF_{i,k}$

ti	GB1	GB2	GB3	GB4	GB5	GB6	GB7	GB8	GB9	GB10	GB11	GB12	r(ti)
0.0027													1.00%
0.0274	100												1.08%
0.0959		106								4.5			1.12%
0.1671											5.5	6	1.16%
0.1808			100										1.16%
0.2575						4		5	7.5				1.21%
0.3397					7								1.25%
0.4247							10						1.29%
0.8521				100									1.60%
1.0959										4.5			1.91%
1.1671											5.5	6	1.95%
1.2575						4		5	7.5				2.03%
1.3397					107								2.09%
1.4247							10						2.11%
2.0959										4.5			2.72%
2.1671											5.5	6	2.73%
2.2603						104		5	7.5				2.75%
2.4274							110						2.82%
3.0986										4.5			3.14%
3.1699											5.5	6	3.14%
3.2603								105	7.5				3.15%
4.0986										4.5			3.51%
4.1699											5.5	6	3.51%
4.2603									107.5				3.52%
5.0986										104.5			4.19%
5.1699											5.5	6	4.19%
6.1699											5.5	6	4.01%
7.1726											105.5	6	4.44%
8.1726												6	4.66%
9.1726												106	4.88%
CPI_0^k	61.28	95.24	62.60	62.60	63.52	87.29	75.46	93.30	83.73	100.60	86.15	85.34	

Optimization with objective constraints
The meaning of R_i

The components of the DYC vector, R_i , are a solution of the following K nonlinear equations.

$$\sum_{i=1}^{N} \frac{CF_{i,k}}{(1+R_i)^{t_i}} = P_m^k \qquad k = 1,\dots,K$$

These are K equations in N unknowns and when $K<N$ there are an infinite number of possible solutions.

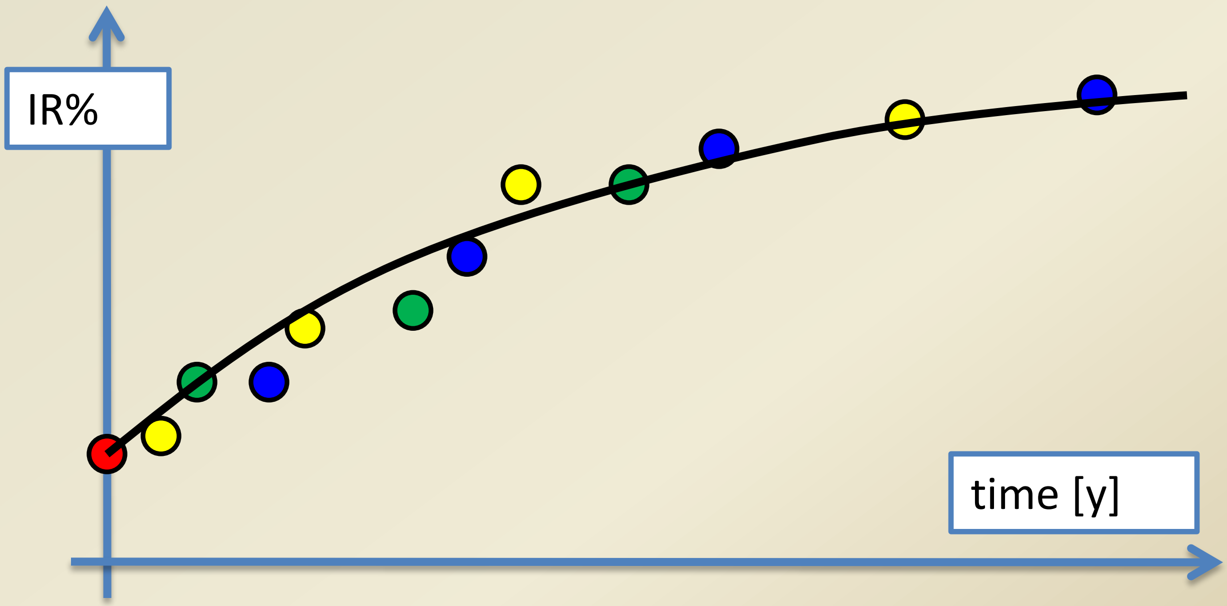
The above equations will be used as **objective constraints** in the optimization procedure.

The target function

The goal is to generate an NSS curve that best fits the DYC vector.

The following target function does the job

$$S_Y(\boldsymbol{\tau}, \boldsymbol{\beta}, \mathbf{R}) = \sum_{i=1}^{N} \left\{ r_{NSS}(\boldsymbol{\tau}, \boldsymbol{\beta}, t_i) - R_i \right\}^2$$



The optimization problem

$$\text{Minimize}\left[S_Y(\boldsymbol{\tau},\boldsymbol{\beta},\mathbf{R})=\sum_{i=1}^{N}\left\{r_{NSS}(\boldsymbol{\tau},\boldsymbol{\beta},t_i)-R_i\right\}^2\right]$$

With respect to $(\boldsymbol{\tau},\boldsymbol{\beta},\mathbf{R})$

Subject to:

1.) $\beta_0+\beta_1=r_a$

2.) $\beta_0>0$

3.) $\boldsymbol{\tau}>\boldsymbol{\tau}^*$

4.) $\sum_{i=1}^{N}\frac{CF_{i,k}}{(1+R_i)^{t_i}}=P_m^k \qquad \forall\{k\in I_k\}$

Comments

1. Constraint No. 1 is required only in the calculation of nominal yield curve
2. There are $6+N$ optimization variables
3. The objective constraints (3) may include only a subset of the traded bonds.
4. The R_i's are auxiliary variables and have no use after the NSS parameters are obtained

Comparison of Methods

Standard (price based) method

$$\text{Minimize}\left[S_P(\boldsymbol{\beta},\boldsymbol{\tau})=\sum_{k=1}^{K}W_k\left(P_m^k-P_{NSS}^k(\boldsymbol{\beta},\boldsymbol{\tau})\right)^2\right]$$

With respect to $(\boldsymbol{\tau},\boldsymbol{\beta})$

$S.T\ldots$

problem with ***6*** variables,

dependent on weights, W_k and

possibly arbitrary constrains

Alternative (yield based) method

$$\text{Minimize}\left[S_Y(\boldsymbol{\tau},\boldsymbol{\beta},\mathbf{R})=\sum_{i=1}^{N}\left\{r_{NSS}(\boldsymbol{\tau},\boldsymbol{\beta},t_i)-R_i\right\}^2\right]$$

With respect to $(\boldsymbol{\tau},\boldsymbol{\beta},\mathbf{R})$

$S.T\ldots\ldots$

- problem with ***6+N*** variables,
- independent of weights
- objective constraints

Dimension of Yield Based Method

If all the bonds pay in the same dates, at frequency ν *(=1, 2, 4, 12)* and there are K bonds with maturities up to $T_{\max}$, then for arbitrary K

$$N = \nu \times T_{\max}$$

In the other extreme we have K $(<365/\nu)$ bonds with non-overlapping payment days so that

$$N = \nu \times K \times T_{\max} < 365 \times T_{\max}$$

Most cases will lay between these two extremes but closer to the lower (first) bound.

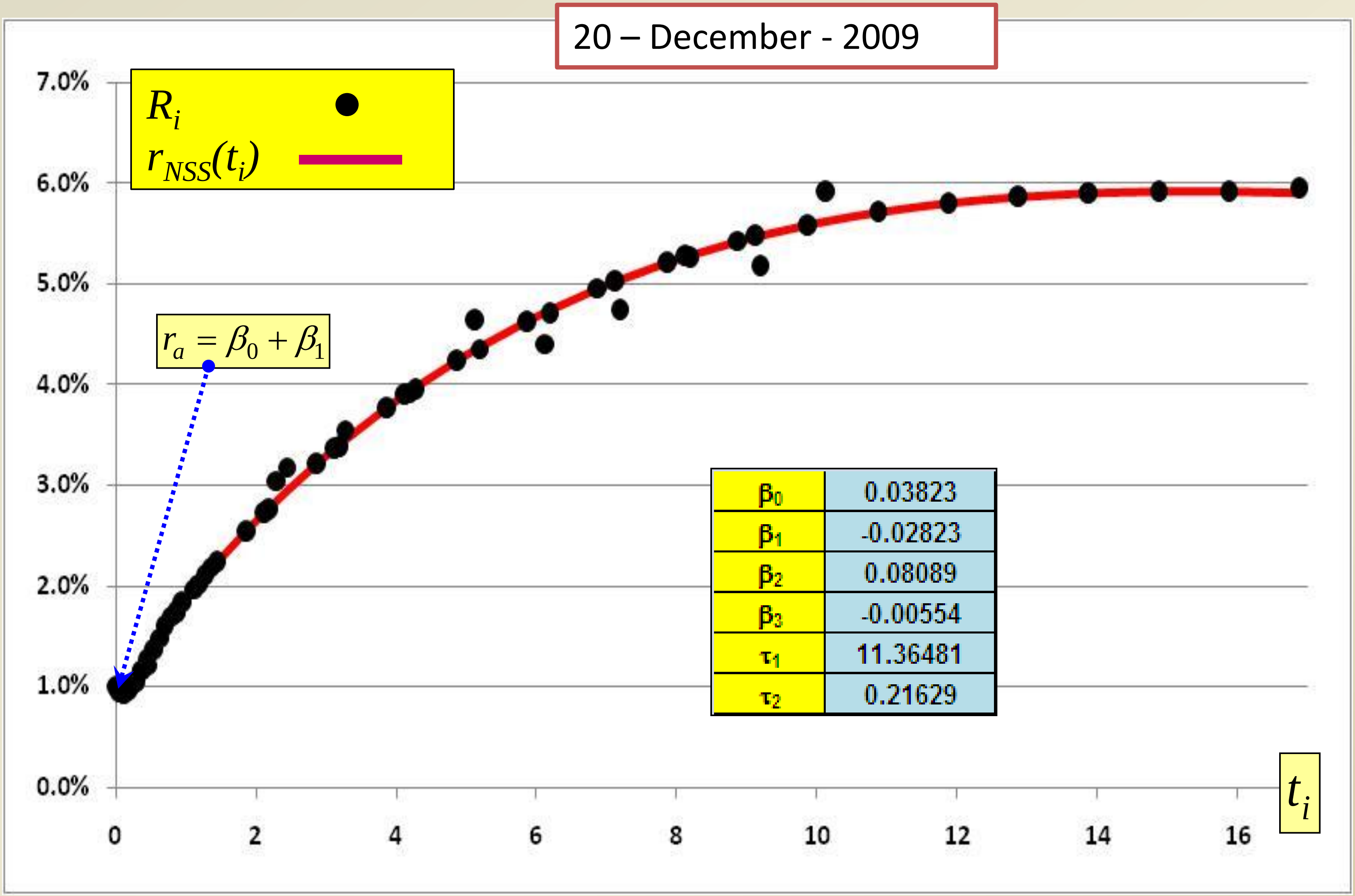
The Israeli Market

As of Dec. 2009

Nominal (cash) bonds	Inflation linked bonds	Parameter
1	1	ν
16.87	26.46	$T_{\max}$ (years)
24	17	K
405	449	$\nu \times K \times T_{\max}$
60	70	N

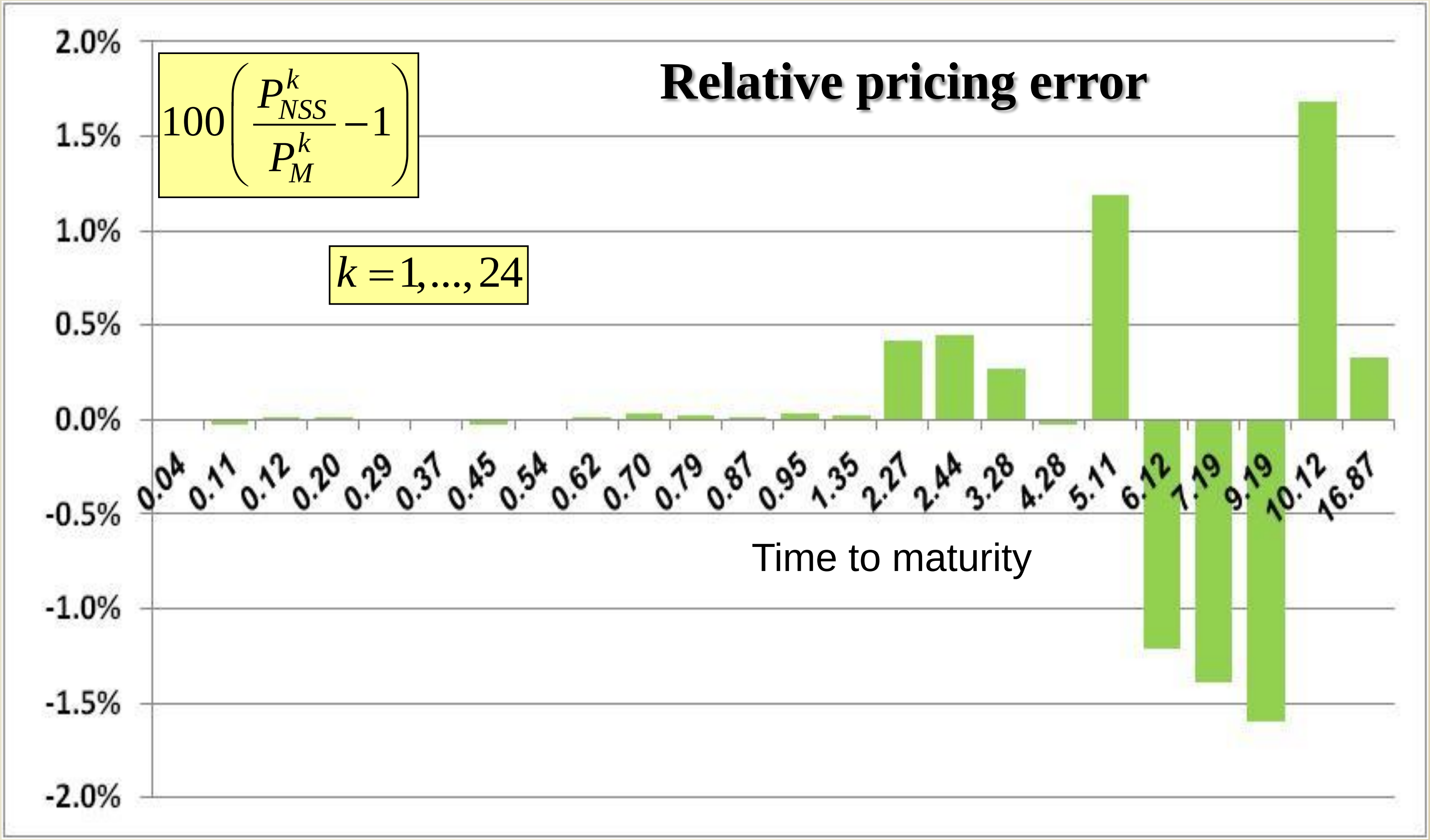
The dimension of an optimization problem is $6+N$ ~ 100 variables with around 25 nonlinear constraints.
This is a small-medium size problem for advanced commercial optimization systems available today.

Example 1 – Nominal Yield Curve

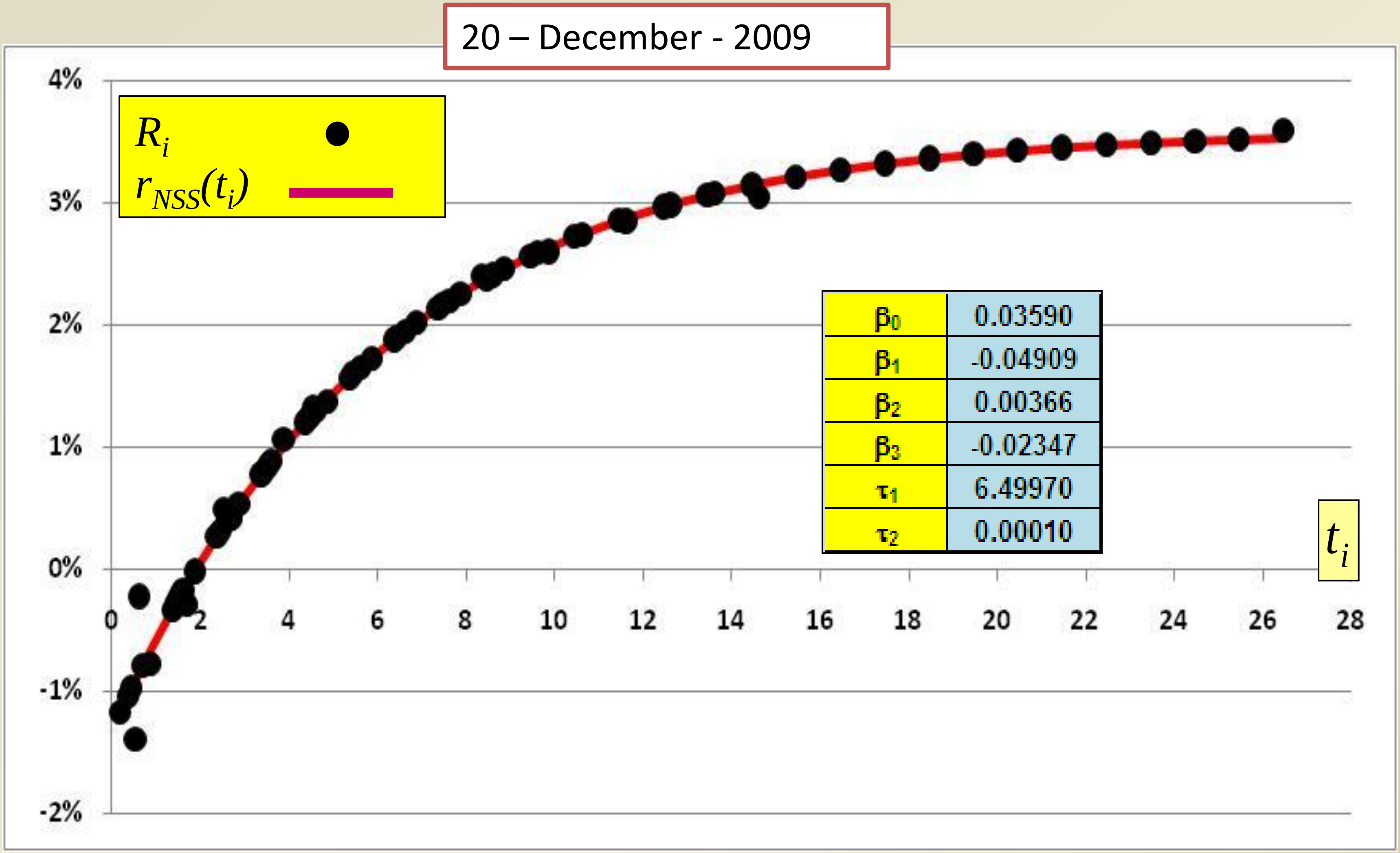


β_0	0.03823
β_1	-0.02823
β_2	0.08089
β_3	-0.00554
τ_1	11.36481
τ_2	0.21629

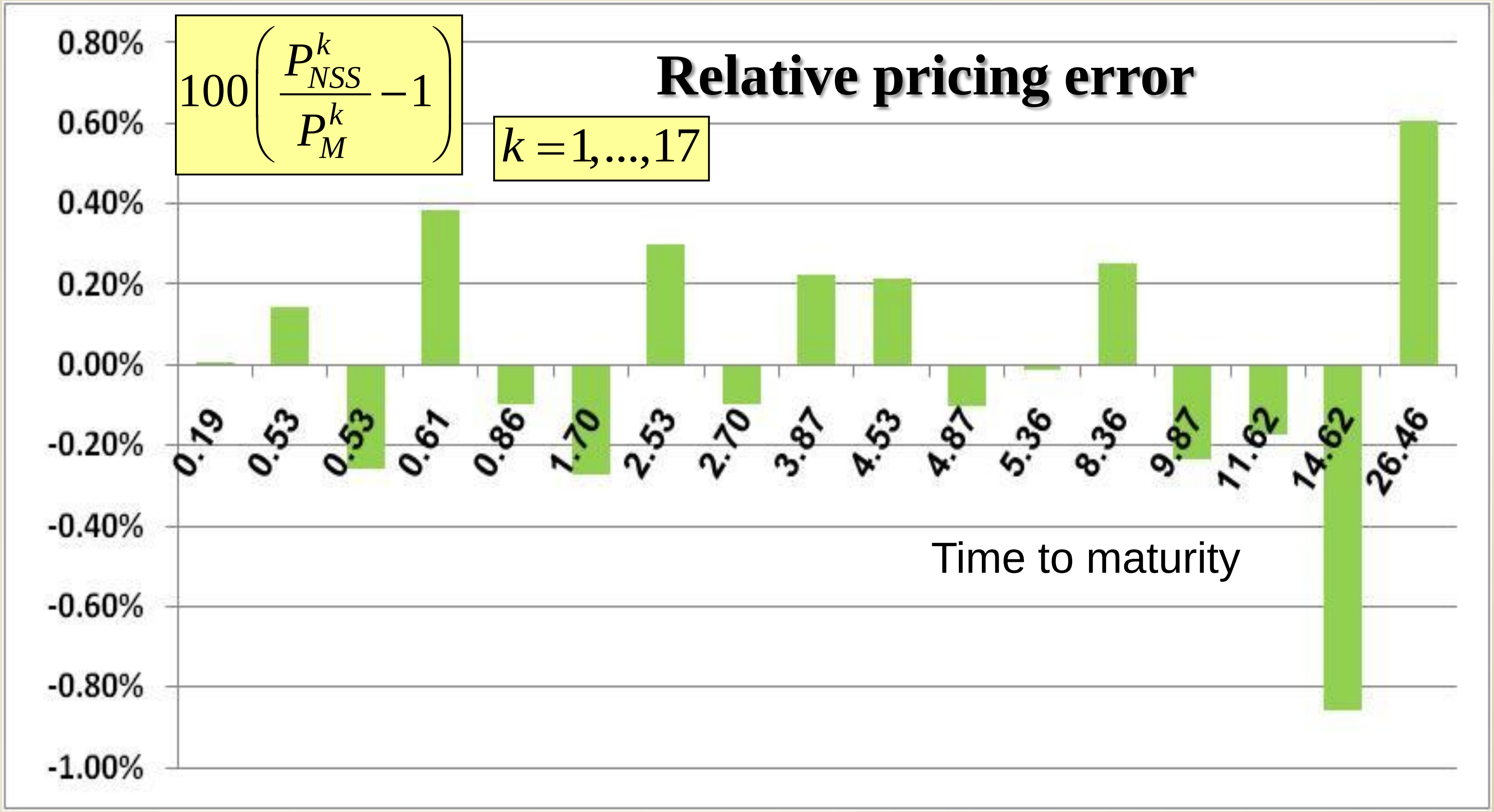
Example 1 – Nominal Yield Curve



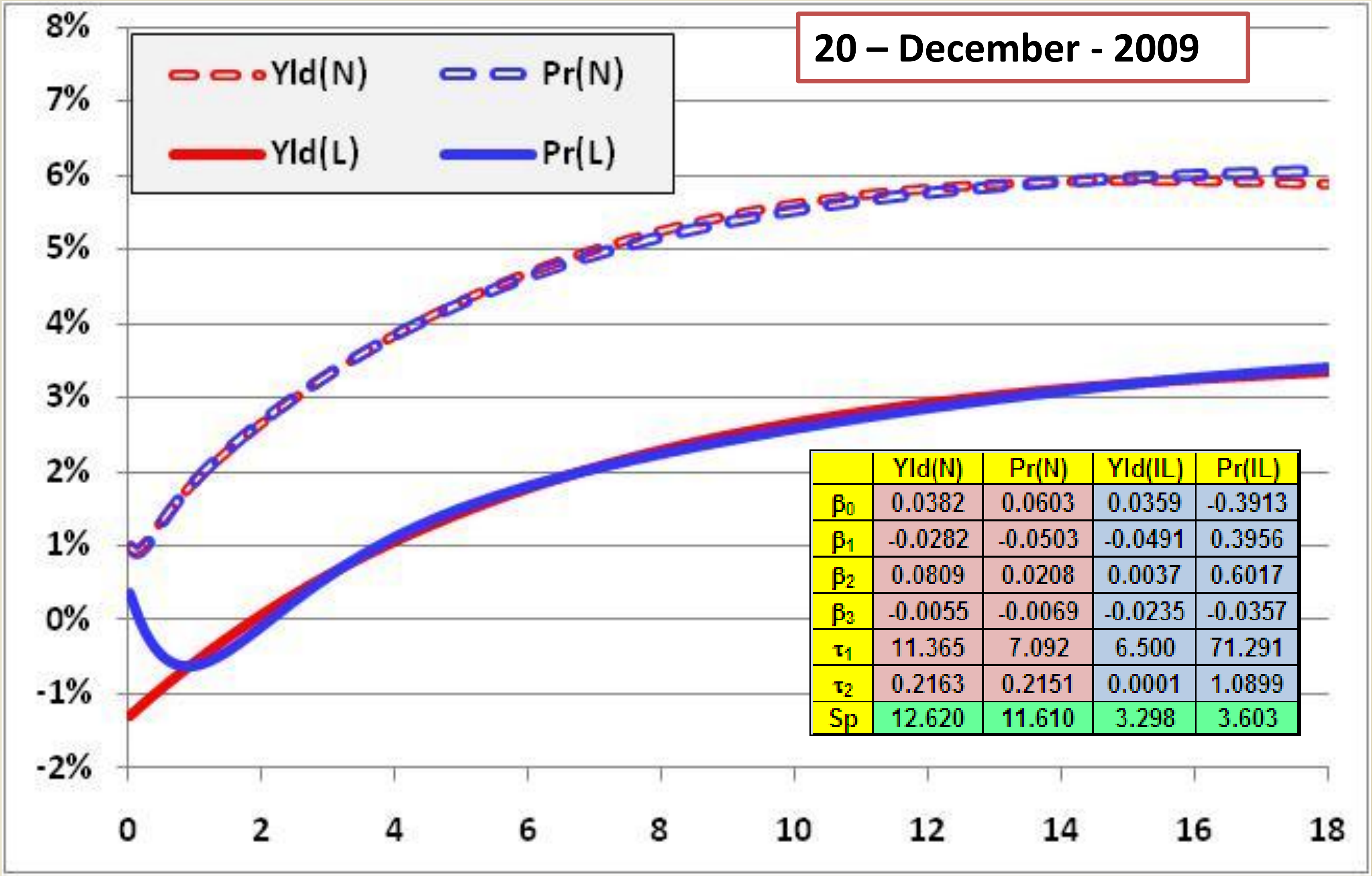
Example 2 – Real Yield Curve



Example 2 – Real Yield Curve

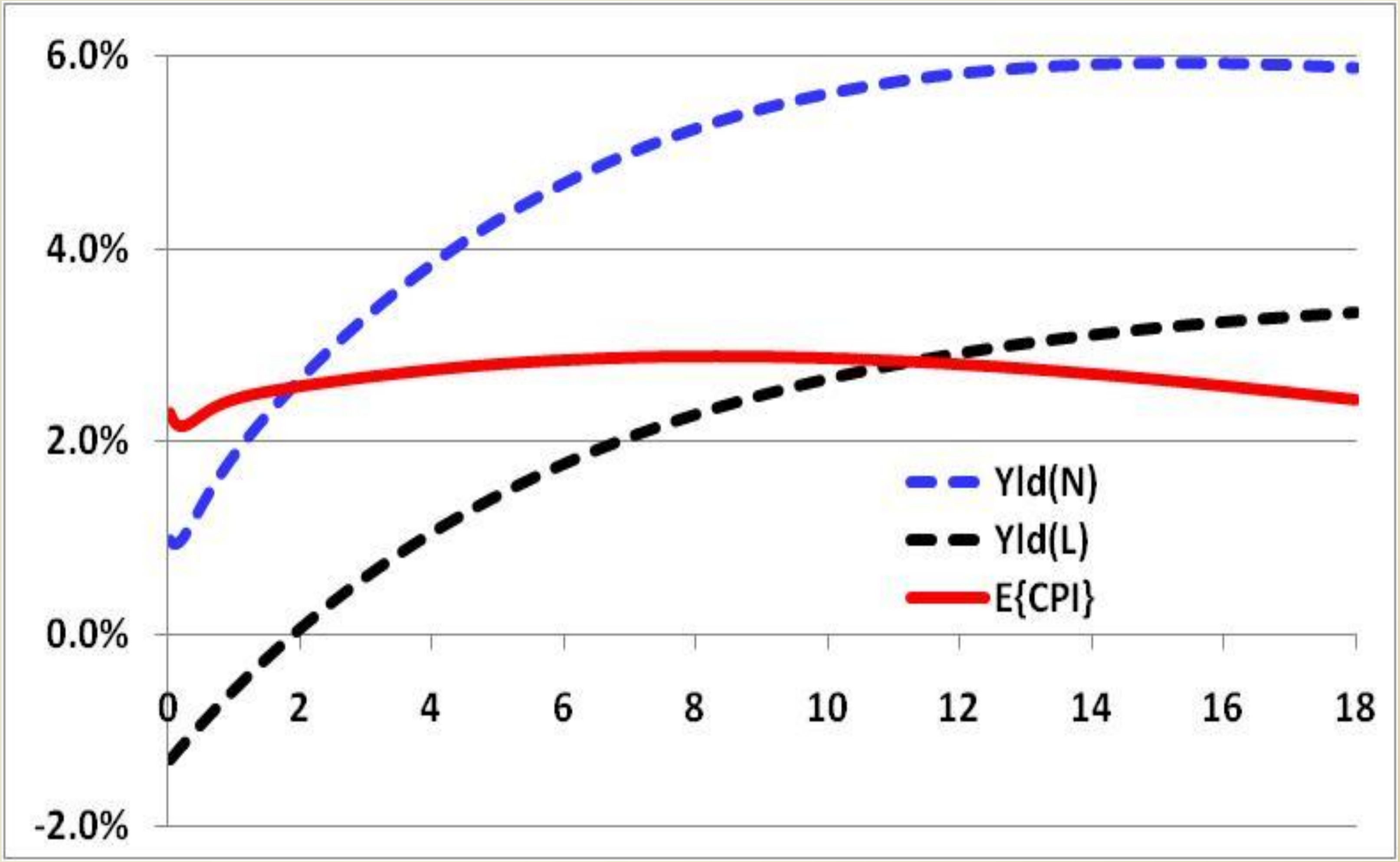


Yield Based vs. Price Based Method



	Yld(N)	Pr(N)	Yld(IL)	Pr(IL)
β_0	0.0382	0.0603	0.0359	-0.3913
β_1	-0.0282	-0.0503	-0.0491	0.3956
β_2	0.0809	0.0208	0.0037	0.6017
β_3	-0.0055	-0.0069	-0.0235	-0.0357
τ_1	11.365	7.092	6.500	71.291
τ_2	0.2163	0.2151	0.0001	1.0899
Sp	12.620	11.610	3.298	3.603

Inflation expectations using Fisher formula



Extensions – Other useful constraints

Another possible (non-binding) objective constraint is the strictly decreasing time dependence of the "time value of money".
This set of N-1 constraints should be:

$$\frac{1}{(1+R_{i+1})^{T_{i+1}}} \leq \frac{1}{(1+R_i)^{T_i}} \qquad i = 1, \ldots, N-1$$

Imposing these constrains has the effect of reducing the roughness of the R_i *vs.* t_i dependence.

These constraints may be too restrictive and impede the realization of the other, binding, constraints.

Extensions – Other useful constraints

When a bond is very liquid and its intraday price volatility is low, the price can be used as anchor (like the zero time anchor $\beta_0+\beta_1=\mathbf{r}_a$) as follows

$$\sum_{i=1}^{N} \frac{CF_i^a}{\left(1+r_{NSS}(\boldsymbol{\beta},\boldsymbol{\tau},t_i)\right)^{t_i}} = P_M^a$$

This constraint is similar to the original one that depends on the R_is

$$\sum_{i=1}^{N} \frac{CF_i^a}{\left(1+R_i\right)^{t_i}} = P_M^a$$

Open question (under study)

Which optimization method is the best, adds up to answering questions

such as how does $S_P(\boldsymbol{\beta}, \boldsymbol{\tau})$ relates to $S_Y(\boldsymbol{\beta}, \boldsymbol{\tau}, \mathbf{R})$

Specifically, do they attain minimum at the same points and under what conditions.

Or what are the convergence advantages/disadvantages of the two methods?

Which method is more accurate?

Summary and conclusions

- Overview of the Israeli government bond market in ILS
- NSS functions are flexible enough to adjust to most yield curve shapes
- Adjusting NSS curve to a bond market can be done by price-based optimization (standard) method or by yield-based optimization methods.
- The yield based method takes advantage of all the market prices through objective constraints.
- Which method is better is an open question to be studied further.

Thank You